

# First Order Differential Equations

**Seperable Equations** A differential equation is called seperable if it is of the form

$$g(y)y' = f(x)$$

An equation is seperable if we can isolate all  $y$  terms on one side of the equation and all  $x$  terms on the other side. Equations of this type can be solved by integrating each side of the equation with respect to the appropriate variable.

## Examples

1.  $y' = yx$

This equation is seperable, as can be seen after dividing by  $y$ . This gives  $\frac{y'}{y} = x$ . Integrating both sides gives  $\ln y = x + C \implies y = e^{x+C} = Ce^x$ . When we divided by  $y$ , we tacitly assumed that  $y \neq 0$ . We must therefore check if  $y = 0$  solves the differential equation. The solutions are then  $y = 0$  and  $y = Ce^x$ .

2.  $2xy^2 - x^4y' = 0$

We can rearrange this equation to give  $\frac{2}{x^3} = \frac{y'}{y^2}$ . This is seperable, and the solution is revealed by integrating.  $\frac{-1}{x^2} + C = \frac{-1}{y} \implies y = \frac{x^2}{1+Cx^2}$ .

**First Order Linear Equations** These differential equations take the general form

$$y' + p(x)y = q(x)$$

where  $p(x)$  and  $q(x)$  are functions of  $x$  only. The following are examples of linear equations.

1.  $y' + x^2y = 0$

2.  $y' + \cos(x)y = x^2$

3.  $y' + \frac{y}{1-x} = e^x$

The following equations would not qualify as linear.

1.  $(y')^2 - \sin(x)y = 0$

2.  $y' + \frac{x^2}{y} = 2x$

3.  $y' + e^xy = y^2$

To solve these equations, we use the integrating factor  $\mu = e^{\int p(x) dx}$ . With this integrating factor, the solution can then be written as  $y = \frac{1}{\mu} \int \mu q(x) dx$ .

### Examples

1.  $y' + \frac{y}{x} = 2e^{x^2}$

In this case,  $p(x) = \frac{1}{x}$  and  $\mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$ . Using our above equation for  $y$  gives the solution  $y = \frac{1}{x} \int 2xe^{x^2} dx = \frac{1}{x}(e^{x^2} + C)$

2.  $y' + y \cos x = \cos x$

In this case,  $p(x) = \cos x$  and  $\mu = e^{\int \cos x dx} = e^{\sin x}$ . Again, applying the solution equation gives  $y = \frac{1}{e^{\sin x}} \int \cos x e^{\sin x} dx = e^{-\sin x}(e^{\sin x} + C) = 1 + Ce^{-\sin x}$

**Exact Equations** An equation of the form

$$M dx + N dy = 0$$

with  $M$  and  $N$  functions of  $x$  and  $y$ , is said to be exact if  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ .

To solve an exact equation, we follow these steps:

1. Our solution will be  $F(x, y) = \Psi(y) + \int M dx = C$ , where  $\Psi(y)$  is a function entirely of  $y$  to be found later.
2. Calculate the integral  $\int M dx$ .
3. Take the derivative of  $F(x, y)$  with respect to  $y$ . Set this equal to  $N$  and solve for  $\Psi'(y)$ .  
 $\Psi'(y) = N - \frac{\partial \int M dx}{\partial y}$ .
4. Find  $\Psi(y)$  by integrating  $\Psi'(y)$  with respect to  $y$ .  $\Psi(y) = \int \Psi'(y) dy$ .
5. Plug  $\Psi(y)$  into  $F(x, y)$  to obtain the solution.

### Examples

1.  $2xy dx + (x^2 + 2y) dy = 0$

Here  $M = 2xy$  and  $N = x^2 + 2y$ . We see the equation is exact since  $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ .  
 $F(x, y) = \int 2xy dx + \Psi(y) = x^2y + \Psi(y)$ . Now we solve for  $\Psi(y)$ .  $\Psi'(y) = N - \frac{\partial(x^2y)}{\partial y} = (x^2 + 2y) - x^2 \implies \Psi'(y) = 2y$ . Integrating we see that  $\Psi(y) = y^2$ . Our solution is then  $x^2y + y^2 = c$ .

2.  $(2xy - 9x^2) dx + (2y + x^2 + 1) dy = 0$

Here  $M = 2xy - 9x^2$  and  $N = 2y + x^2 + 1$ . We see the equation is exact since  $\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}$ .  $F(x, y) = \int 2xy - 9x^2 dx + \Psi(y) = x^2y - 3x^3 + \Psi(y)$ . Next, solve for  $\Psi(y)$ .  $\Psi'(y) = N - \frac{\partial(x^2y - 3x^3)}{\partial y} = (2y + x^2 + 1) - x^2 = 2y + 1$ . Integrate this to see that  $\Psi(y) = y^2 + y$ . The solution is then  $F(x, y) = x^2y - 3x^3 + y^2 + y = C$ .

**Making Equations Exact** Occasionally, one will encounter an equation of the form

$$M dx + N dy = 0$$

that does not meet the criterion for exactness. In certain situations, we can find an appropriate integrating factor which will transform this into an exact equation.

Case 1 Integrating factors of  $x$  only: If the quantity  $p(x) = \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$  is a function with no occurrences of  $y$ , then  $\mu = e^{\int p(x) dx}$  is an integrating factor for the differential equation.

Case 2 Integrating factors of  $y$  only: If the quantity  $p(y) = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$  is a function with no occurrences of  $x$ , then  $\mu = e^{\int p(y) dy}$  is an integrating factor for the differential equation.

When the integrating factor  $\mu$  exists, one may multiply the differential equation by  $\mu$  to create an exact equation.

### Examples

1.  $(y^2(x^2 + 1) + xy) dx + (2xy + 1) dy = 0$

$\frac{\partial M}{\partial y} = 2y(x^2 + 1) + x$ , and  $\frac{\partial N}{\partial x} = 2y$ . As we can see, this equation is not exact. We will search for an integrating factor.  $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y(x^2+1)+x-2y}{2yx+1} = \frac{2yx^2+x}{2yx+1} = x$ . This is a function entirely of  $x$  so that  $\mu = e^{\int x dx} = e^{\frac{x^2}{2}}$  will be an integrating factor.

Multiply the initial equation by  $\mu$  to give  $(e^{\frac{x^2}{2}}y^2(x^2 + 1) + e^{\frac{x^2}{2}}xy) dx + (2e^{\frac{x^2}{2}}xy + e^{\frac{x^2}{2}}) dy = 0$ . Now  $\frac{\partial M}{\partial y} = 2x^2e^{\frac{x^2}{2}}y + 2ye^{\frac{x^2}{2}} + xe^{\frac{x^2}{2}} = \frac{\partial N}{\partial x}$  so that the equation is now exact and can be solved via the methods previously discussed.

2.  $(x^2y + 2y^2 \sin x) dx + (\frac{2}{3}x^3 - 6y \cos x) dy = 0$

The equation is not exact since  $\frac{\partial M}{\partial y} = x^2 + 4y \sin x$ , and  $\frac{\partial N}{\partial x} = 2x^3 + 6y \sin x$ . Now attempt to find an integrating factor.  $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x^3 + 6y \sin x - x^2 - 4y \sin x}{x^2y + 2y^2 \sin x} = \frac{x^2 + 2y \sin x}{x^2y + 2y^2 \sin x} = \frac{1}{y}$ .

This is a function entirely of  $y$  so the equation has an integrating factor of the form  $e^{\int \frac{1}{y} dy} = e^{\ln y} = y$ .

Multiply the initial equation by  $y$  to give  $(x^2y^2 + 2y^3 \sin x) dx + (\frac{2}{3}x^3y - 6y^2 \cos x) dy = 0$ . Now  $\frac{\partial M}{\partial y} = 2x^2y + 6y^2 \sin x = \frac{\partial N}{\partial x}$ . As we can see, this equation is now exact and can be solved accordingly.